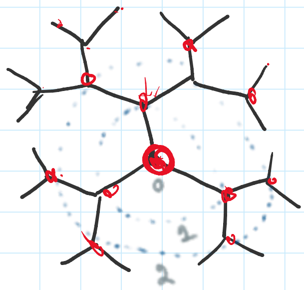
Ising model on Bethe Lattice

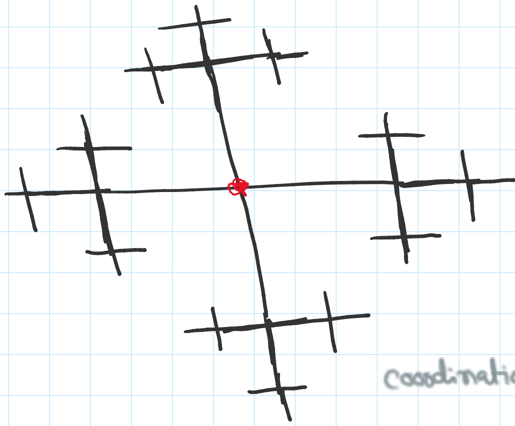
Saturday, May 8, 2021   10:50 AM

Ref: Book of Baxter, Exactly solvable models

Caley tree



of coordination number $q=3$



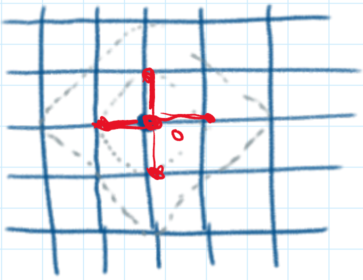
coordination $q=4$

at nth generation there are $q(q-1)^{n-1}$ points for $n \geq 1$

In a tree of L-th generation, total number of nodes $N = 1 + q\sum_{n=1}^{L}(q-1)^{n-1} = \frac{q(q-1)^L - 2}{q-2}$

Dimension: One measure of dimension is how the number of nodes grow with distance (generation).

regular lattice



nodes $\sim n^d \Rightarrow d = \lim_{n\to\infty} \frac{\log N}{\log n}$

Caley tree

$$d = \lim_{n\to\infty} \frac{\ln N}{\ln n} = \lim_{n\to\infty} \frac{n \ln(q-1)}{\ln n} = \infty \quad \text{for } q>2.$$

Caley tree is infinite dimensional. for $q>2$.

Important difference

Boundary nodes have only one neighbor. However, number of boundary nodes is of same order as bulk nodes. Therefore, to get bulk properties we must not consider the boundary nodes.

$$\text{Caley tree} \xrightarrow[\text{Boundary removed}]{} \text{Bethe lattice}$$

Also means $Z_{caley} \neq Z_{Bethe}$.

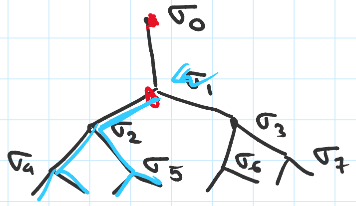
But we can define thermodynamic limit if at the central node $\lim_{L\to\infty} m_{caley}$ converge. The limiting $m = m_{Bethe}$. In a similar way we can define free energy density on Bethe lattice

Ising model partition function on caley tree.

$$Z_L = \sum_{\{\sigma_i\}} e^{+\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j}$$

Only nearest neighbor interaction.
Zero magnetic field.

Because there are no loops on a tree, there is a recursion relation.
Consider a branch



Define $Q_L(\sigma_0|\bar{\sigma}') = e^{\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j}$

all spins in the children branches

Then,

$$\boxed{Q_L(\sigma_0|\bar{\sigma}') = e^{\beta \sigma_0 \sigma_1} \prod_{k=1}^{q-1} Q_{L-1}(\sigma_1|\bar{\sigma}_k'')} \rightarrow (1)$$

all spins in the kth sub branch.

In terms of Q, the partition function

$$Z_L = \sum_{\sigma_0} \Big[\sum_{\bar{\sigma}'} Q_L(\sigma_0|\bar{\sigma}') \Big]^q = -\sum_{\sigma} g_L(\sigma_0))^q = g_L(+)^q + g_L(-)^q$$

magnetization at the zeroth node

$$m_0 = \frac{1}{Z_L} \sum_{\sigma_0} \sigma_0 [g_L(\sigma_0)]^q = \frac{g_L(+ - g_L(-)^q}{g_L(+) + g_L(-)^q} = \frac{1 - x_L^q}{1 + x_L^q} \quad \text{here } x_L = \frac{g_L(-)}{g_L(+)}$$

Here we defined

$$g_L(\sigma_0) = \sum_{\bar{\sigma}'} Q_L(\sigma_0|\bar{\sigma}') = \sum_{\sigma_1} e^{\beta\sigma_0\sigma_1} \prod_{k=1}^{q-1} \sum_{\sigma_k''} Q_{L-1}(\sigma_1|\bar{\sigma}_k'')$$

$$= \sum_{\sigma_1} e^{\beta\sigma_0\sigma_1} [g_{L-1}(\sigma_1)]^{q-1}$$

$$\Rightarrow \boxed{g_L(\sigma_0) = e^{\beta\sigma_0}[g_{L-1}(+)]^{q-1} + e^{-\beta\sigma_0}[g_{L-1}(-)]^{q-1}}$$

This gives

$$\left.\begin{aligned} g_L(+) &= e^{\beta}(g_{L-1}(+))^{q-1} + e^{-\beta}[g_{L-1}(-)]^{q-1} \\ g_L(-) &= e^{-\beta}[g_{L-1}(+)]^{q-1} + e^{\beta}[g_{L-1}(-)]^{q-1} \end{aligned}\right\} \Rightarrow \boxed{x_L = \frac{e^{-\beta} + e^{\beta} \cdot x_{L-1}^{q-1}}{e^{\beta} + e^{-\beta} x_{L-1}^{q-1}}}$$

iterate this with initial value

$$x_1 = \frac{g_1(-)}{g_1(+)} = 1 \quad \text{because} \quad g_1(\sigma_0) = \sum_{\sigma_1} e^{\beta\sigma_0\sigma_1} = e^{\beta\sigma_0} + e^{-\beta\sigma_0}$$

The $L \to \infty$ iteration x_∞ gives the magnetization $\boxed{m_0 = \frac{1 - x_\infty^q}{1 + x_\infty^q}}$

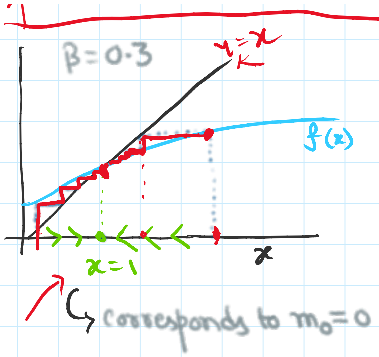
The fixed point:

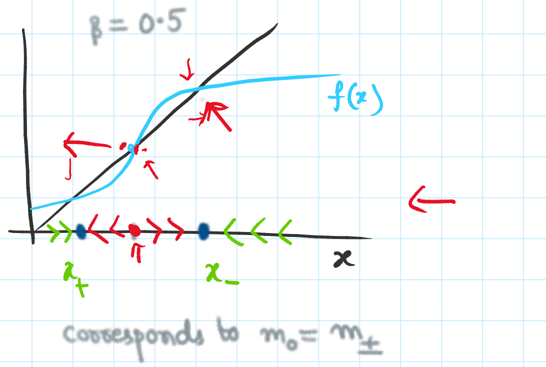
$$\boxed{x_L = \frac{1 + e^{2\beta} \cdot x_{L-1}^{q-1}}{e^{2\beta} + x_{L-1}^{q-1}} = f(x_{L-1})}$$

$x = f(x)$

For $q = 4$

For $q=4$





Remark: The fixed points are solution of

$$x = \frac{1+e^{2\beta}x^{q-1}}{e^{2\beta}+x^{q-1}} \quad \text{for } x\in \text{Real}$$

From the solution, the magnetization can be written as

$$m_0 = \frac{1-\lambda^2}{1+\lambda^2+2\lambda e^{-2\beta}} \quad \text{with} \quad \lambda = \left(\frac{\lambda+e^{-2\beta}}{1+\lambda e^{-2\beta}}\right)^{q-1}$$

This is an example of a parametric form of a solution. Many results, particularly in non-equilibrium stat mech are in parametric form.

Critical temperature:

$$\boxed{\beta_c = \frac{1}{2}\log\frac{q}{q-2}}$$

This is derived by studying when a real solution for fixed point other that $x=1$, starts appearing.

This result shows that there is no phase transition at finite non-zero temperature for a Bethe lattice with coordination number $q\le 2$ [$q=2$ is 1-d lattice]

Free energy: $F_L^{\text{Caley}} = -\frac{1}{\beta}\log Z_L$ for large L does not give free energy on Bethe lattice because boundary nodes are as comparable as bulk nodes.

The free energy on Bethe lattice can be computed following way. Consider an additional magnetic field. The use the stat mech relation

$$\boxed{\frac{\partial}{\partial h}f(h) = -m_0(h)}$$

free energy density

If we consider $m_0(h)$ as the magnetization at 0^{th} site in the thermodynamic limit, then $f(h)$ is the free energy density of Bethe lattice.

In presence of magnetic field h, fixed point equation changes

$$\boxed{\frac{e^{2\beta}-x}{e^{2\beta}x-1}\,x^{q-1} = e^{2h}} \quad \text{and the} \quad \boxed{m_0(h) = \frac{e^{2h}-x^q}{e^{2h}+x^q}}$$

$$\frac{}{e^{2\beta}x-1}\; x \;=\; e \qquad \text{and the } m_0(h) = \frac{}{e^{2h}+x^{q}}$$

Using this $m_0(h)$, and integrating gives

$$f(h) = f(h^*) + \int_h^{h^*} dy\, m_0(y)$$

for h^* large all spins are up, $\Rightarrow$ $f(h^*) = -N_e \cdot \beta - h^* \cdot N$

Total # of edges — Total # of nodes

$$\Rightarrow f(h) = -\beta N_e - h^* N + \int_h^{h^*} dy\, m_0(y)$$

$$= -\beta N_e - hN + \int_h^{h^*} dy\,[m_0(y) - 1] \quad \text{for large } h^*.$$

Finally, writing $h^* \to \infty$, gives the free energy density

$$\boxed{f(h) = -\beta N_e - hN + \int_h^{\infty} dy\,[m_0(y) - 1]}$$

Remark: The free energy can be written at best in a parametric form.

$$\boxed{f(h) = G(x) \quad \text{with} \quad h = \frac{1}{2}\log\left[x^{q-1}\,\frac{e^{2\beta}-x}{e^{2\beta}x-1}\right]}$$

$G(x)$ is given in the book of Baxter, Page 56, eq (4.6.8)

Remark: Critical properties of Ising model on Bethe lattice is similar as for meanfield solution of Ising model. This is partly because Bethe lattice is infinite dimensional.

The critical exponents are same as in mean-field Ising model.

$$\boxed{\alpha = 0\,,\quad \beta = 1/2\,,\quad \gamma = 1\,,\quad \delta = 3}$$

Revision of critical exponents: near continuous transition

Specific heat	$C \sim \lvert T-T_c\rvert^{-\alpha}$	for $h=0$
magnetization	$m \sim (T_c - T)^{\beta}$	for $h=0$ and $T<T_c$
susceptibility	$\chi \sim \lvert T-T_c\rvert^{-\gamma}$	for $h=0$
magnetization	$\lvert m\rvert \sim \lvert h\rvert^{1/\delta}$	for $T=T_c$ and small h

Additional: near critical point, spatial correlation of spins

$$\langle \sigma_r \cdot \sigma_0 \rangle_c \simeq \frac{1}{r^{d-2+\eta}} \cdot e^{-r/\xi}$$

where $\xi \sim \lvert T-T_c\rvert^{-\nu}$ is correlation length.

Scaling laws

These exponents are related

$$\alpha + 2\beta + \gamma = 2$$

$$\alpha + \beta\delta + \beta = 2$$

$$\nu(2-\eta) = \gamma$$

$$\alpha + \nu d = 2$$

These come as a result of thermodynamic constraints, such as fluctuation dissipation relation.

Remark: Inspite of similarity of critical properties with meanfield solution of Ising model, Bethe lattice calculation is closer to a "reality". It has only nearest neighbour interactions, and does not show phase transition in 1d.